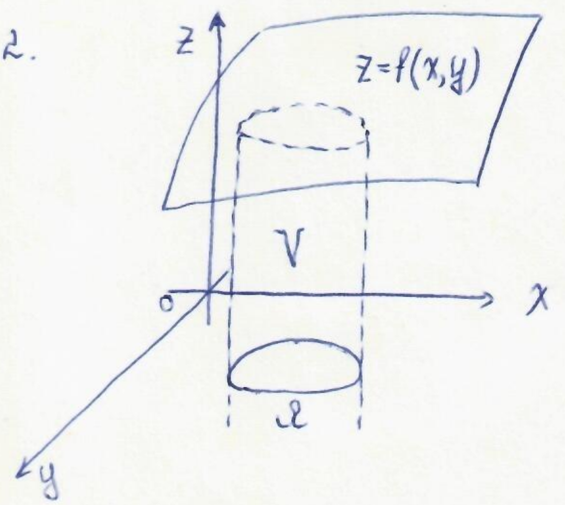
Раздел 2. Тема 1. Занятие 4.

1. Площадь области Ω, расположенной в плоскости OXY дается формулой

$$S = \iint\limits_{\Omega} dxdy$$

2. 

Объем цилиндроида, ограниченного сверху непрерывной поверхностью $z=f(x,y)$, снизу плоскостью $z=0$ и с боков прямой цилиндрической поверхностью, вырезающей на плоскости OXY область Ω, равен

$$V = \iint\limits_{\Omega} f(x,y)\,dxdy.$$

№№ 3986, 3987, 3998, 3995, 4009, 4015, 4020, 4029.

№3986 Найти площадь ограниченную кривой

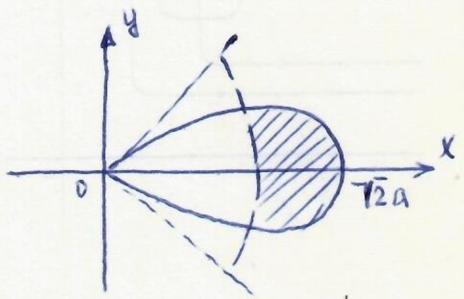
$$(x-y)^2+y^2=a^2 \quad (a>0).$$

Замена переменных $\begin{cases} u=x \\ v=y-x \end{cases} \Rightarrow \begin{cases} x=u \\ y=u+v \end{cases} \Rightarrow I=1.$

$$S=\iint\limits_{\Omega} dx\,dy=\iint\limits_{\Omega'} du\,dv=\iint\limits_{u^2+v^2\le a^2} du\,dv=\pi a^2.$$

№3987. Вычислить площадь, ограниченную кривыми:

$$\left[\begin{array}{l} (x^2+y^2)^2=2a^2(x^2-y^2) \\ x^2+y^2\geqslant a^2 \end{array}\right.$$



два лепестка!

$$\begin{cases} x=r\cos\varphi \\ y=r\sin\varphi \end{cases} \Rightarrow \left[\begin{array}{l} r^2=2a^2\cos 2\varphi \\ r\geqslant a \end{array}\right.$$

$\cos 2\varphi\geqslant 0 \Rightarrow -\frac{\pi}{4}\le\varphi\le\frac{\pi}{4}$ для первого уравнения

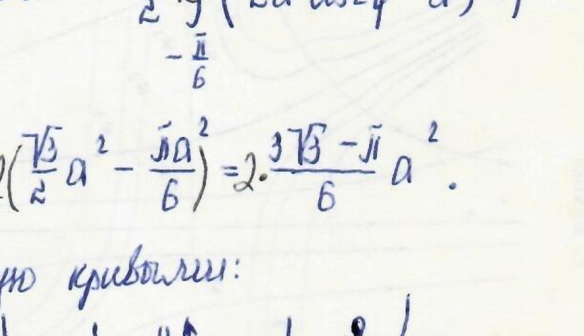
$$\begin{cases} r=a \\ r=a\sqrt{2\cos 2\varphi} \end{cases} \Rightarrow \cos 2\varphi=\frac{1}{2} \Rightarrow \varphi_0=\frac{\pi}{6} \Rightarrow -\frac{\pi}{6}\le\varphi\le\frac{\pi}{6} \Rightarrow$$

$$\begin{cases} -\frac{\pi}{6}\le\varphi\le\frac{\pi}{6} \\ a\le r\le a\sqrt{2\cos 2\varphi} \end{cases} \Rightarrow S=2\int\limits_{-\frac{\pi}{6}}^{\frac{\pi}{6}}d\varphi\int\limits_{a}^{a\sqrt{2\cos 2\varphi}} r\,dr=\frac{1}{2}\cdot 2\int\limits_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\left(2a^2\cos 2\varphi-a^2\right)d\varphi=$$

$$=2\left(\frac{a^2}{2}\sin 2\varphi\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}}-\frac{a^2}{2}\cdot\frac{2\pi}{6}\right)=2\left(a^2\sin\frac{\pi}{3}-\frac{\pi a^2}{6}\right)=2\left(\frac{\sqrt{3}}{2}a^2-\frac{\pi a^2}{6}\right)=2\cdot\frac{3\sqrt{3}-\pi}{6}a^2.$$

№3998 Найти площадь фигуры, ограниченную кривыми:

$$\begin{cases} y^2=2px \qquad x^2=2ry \\ y^2=2qx \qquad x^2=2sy \\ 0<p<q \qquad 0<r<s \end{cases}$$



Замена переменных:

$$u=\frac{y^2}{2x} \quad v=\frac{x^2}{2y} \Rightarrow \begin{cases} p\le u\le q \\ r\le v\le s \end{cases}$$

$$\begin{cases} 2xu = y^2 \\ 2yv = x^2 \end{cases} \Rightarrow \left.\begin{matrix} 4x^2u^2 = y^4 \\ 2yv = x^2 \end{matrix}\right\} \Rightarrow 8u^2v = y^3 \Rightarrow y = 2\sqrt[3]{u^2v}.$$

$$\left.\begin{matrix} 2xu = y^2 \\ 4y^2v^2 = x^4 \end{matrix}\right\} \Rightarrow 8uv^2 = x^3 \Rightarrow x = 2\sqrt[3]{uv^2}.$$

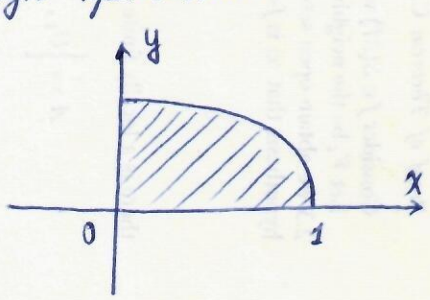
$$\frac{\partial x}{\partial u} = \frac{2}{3}\left(\frac{v}{u}\right)^{\frac{2}{3}} \qquad \frac{\partial x}{\partial v} = \frac{4}{3}\left(\frac{u}{v}\right)^{\frac{1}{3}}$$

$$\frac{\partial y}{\partial u} = \frac{4}{3}\left(\frac{v}{u}\right)^{\frac{1}{3}} \qquad \frac{\partial y}{\partial v} = \frac{2}{3}\left(\frac{u}{v}\right)^{\frac{2}{3}} \Rightarrow I = \frac{4}{9} - \frac{16}{9} = -\frac{4}{3} \Rightarrow |I| = \frac{4}{3} \Rightarrow$$

$$S = \iint\limits_{\Omega} dxdy = \frac{4}{3}\int_p^q du \int_r^s dv = \frac{4}{3}(q-p)(s-r).$$

N3995 Вычислить площадь фигуры, ограниченную кривыми:

$$\begin{cases} \sqrt[4]{\frac{x}{a}} + \sqrt[4]{\frac{y}{b}} = 1 \\ x = 0,\ y = 0 \end{cases}$$



Замена переменных:

$$\begin{cases} x = ar\cos^8\varphi \\ y = br\sin^8\varphi \end{cases} \Rightarrow I = 8abr\cos^7\varphi\sin^7\varphi \Rightarrow$$

$$S = \iint\limits_{\Omega} dxdy = \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 8abr\cos^7\varphi\sin^7\varphi\, d\varphi = 8ab\int_0^{\frac{\pi}{2}}\cos^7\varphi\sin^7\varphi\, d\varphi \int_0^1 r\,dr =$$

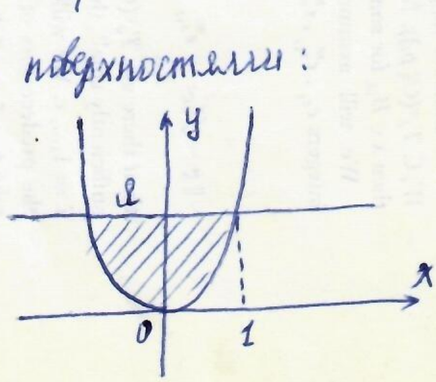
$$= 4ab\int_0^{\frac{\pi}{2}}\cos^7\varphi\sin^7\varphi\, d\varphi = \frac{ab}{32}\int_0^{\frac{\pi}{2}}\sin^7 2\varphi\, d\varphi = \frac{ab}{64}\int_0^{\pi}\sin^7 u\, du =$$

$$= \frac{ab}{64}\int_0^{\pi}\sin^6 u\,\sin u\, du = -\frac{ab}{64}\int_0^{\pi}(1-\cos^2 u)^3 d(\cos u) = \left| z = \cos u \right| =$$

$$= \frac{ab}{64}\int_{-1}^{1}(1-z^2)^3 dz = \frac{ab}{64}\left\{ z - z^3 + \frac{3}{5}z^5 - \frac{z^7}{7} \right\}\Big|_{-1}^{1} = \frac{ab}{64}\cdot\frac{32}{35} = \frac{ab}{70}.$$

N4009. Найти объем фигуры, ограниченную поверхностями:

$$\begin{cases} z = x^2 + y^2 \\ y = x^2 \\ y = 1 \\ z = 0 \end{cases}$$



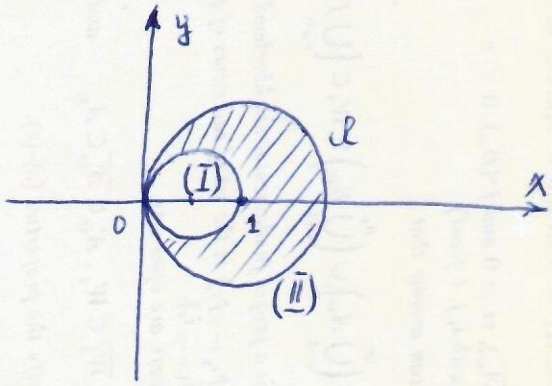
$$V = \iint\limits_{\Omega} (x^2+y^2)\,dx\,dy = 2\int_0^1 dx \int_{x^2}^{1} (x^2+y^2)\,dy = 2\int_0^1 dx \left(x^2 y + \frac{y^3}{3} \right)\Bigg|_{x^2}^{1} =$$

$$= 2\int_0^1 \left(x^2 + \frac{1}{3} - x^4 - \frac{x^6}{3} \right) dx = 2\left(\frac{x^3}{3} + \frac{1}{3}x - \frac{x^5}{5} - \frac{x^7}{21} \right)\Bigg|_0^1 =$$

$$= 2\left(\frac{2}{3} - \frac{1}{5} - \frac{1}{21} \right) = 2\cdot\frac{70-26}{105} = \frac{2\cdot 44}{105} = \frac{88}{105}.$$

№4015 Найти объем фигуры, ограниченной поверхностями:

$$\left[\begin{array}{l} z = x^2+y^2 \\ x^2+y^2 = x \\ x^2+y^2 = 2x \\ z = 0 \end{array}\right.$$



$$\left\{\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array}\right. \Rightarrow \left\{\begin{array}{l} -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le \cos\varphi \end{array}\right. \quad (I)$$

$$\left\{\begin{array}{l} -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le 2\cos\varphi \end{array}\right. \quad (II)$$

$$V = \iint\limits_{\Omega} (x^2+y^2)\,dx\,dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^3\,dr - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\cos\varphi} r^3\,dr = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 16\cos^4\varphi\,d\varphi -$$

$$- \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\varphi\,d\varphi = \frac{15}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\varphi\,d\varphi = \frac{15}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + 2\cos 2\varphi + \cos^2 2\varphi)\,d\varphi =$$

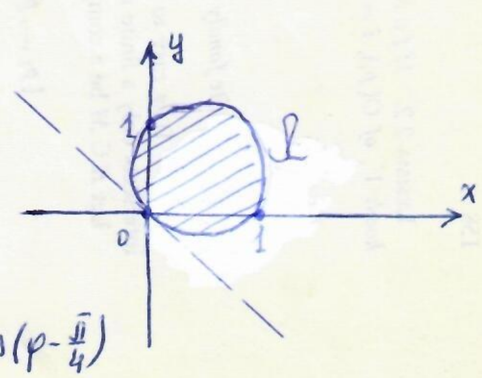
$$= \frac{15}{16}\pi + \frac{15}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos 2\varphi\,d\varphi + \frac{15}{32}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos 4\varphi)\,d\varphi = \frac{15}{16}\pi + \frac{15}{16}\cancelto{0}{\sin 2\varphi}\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{15}{32}\pi +$$

$$+ \frac{15}{128}\cancelto{0}{\sin 4\varphi}\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{15}{16}\pi + \frac{15}{32}\pi = \frac{45}{32}\pi.$$

№4020 Найти объем фигуры, ограниченной поверхностями:

$$\left[\begin{array}{l} z = x^2+y^2 \\ z = x+y \end{array}\right. \Rightarrow x^2+y^2 = x+y \Rightarrow$$

$$\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2$$



Замена переменных: $\left\{\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array}\right. \Rightarrow \begin{array}{l} -\frac{\pi}{4} \le \varphi \le \frac{3\pi}{4} \\ 0 \le r \le \sqrt{2}\cos\left(\varphi - \frac{\pi}{4}\right) \end{array}$

$$\begin{cases} z = x^2+y^2 = r^2 \\ z = x+y = \sqrt{2}\, r\cos(\varphi-\frac{\pi}{4}) \end{cases} \Rightarrow \quad V = \iint\limits_{\Omega}(x+y)\,dx\,dy - \iint\limits_{\Omega}(x^2+y^2)\,dx\,dy = \iint\limits_{\Omega}(x+y-x^2-y^2)\,dx\,dy =$$

$$= \int\limits_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \int\limits_0^{\sqrt{2}\cos(\varphi-\frac{\pi}{4})} \left(\sqrt{2}\, r\cos(\varphi-\tfrac{\pi}{4}) - r^2\right) r\,dr = \int\limits_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \int\limits_0^{\sqrt{2}\cos(\varphi-\frac{\pi}{4})} \left(\sqrt{2}\, r^2\cos(\varphi-\tfrac{\pi}{4}) - r^3\right) dr =$$

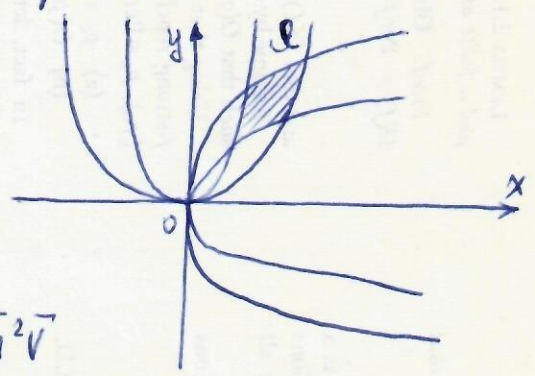
$$= \int\limits_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\frac{\sqrt{2}}{3} r^3\cos(\varphi-\tfrac{\pi}{4}) - \frac{1}{4}r^4\right)\Bigg|_0^{\sqrt{2}\cos(\varphi-\frac{\pi}{4})} d\varphi = \int\limits_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\frac{\sqrt{2}}{3}\cdot 2\sqrt{2}\cos^4(\varphi-\tfrac{\pi}{4}) - \frac{1}{4}\cdot 4\cos^4(\varphi-\tfrac{\pi}{4})\right) d\varphi =$$

$$= \int\limits_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{3}\cos^4(\varphi-\tfrac{\pi}{4})\,d\varphi = \left\{\psi = \varphi - \frac{\pi}{4}\right\} = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3}\cos^4\psi\,d\psi = \frac{2}{3}\int\limits_0^{\frac{\pi}{2}}\cos^4\psi\,d\psi = \frac{1}{6}\int\limits_0^{\frac{\pi}{2}}\left(1 + \overset{0}{\cancel{\cos 2\psi}} + \overset{0}{\cancel{\frac{1+\cos 4\psi}{2}}}\right) d\psi$$

$$= \frac{1}{6}\cdot\frac{3}{2}\cdot\frac{\pi}{2} = \frac{\pi}{8}.$$

№4029. Найти объем фигуры, ограниченной поверхностями:

$$\begin{cases} z = xy \\ x^2 = y \\ x^2 = 2y \end{cases} \qquad \begin{matrix} y^2 = x \\ y^2 = 2x \\ z = 0 \end{matrix}$$



Замена переменных

$$\begin{cases} u = \frac{x^2}{y} \\ v = \frac{y^2}{x} \end{cases} \Rightarrow \begin{cases} 1 \le u \le 2 \\ 1 \le v \le 2 \end{cases} \Rightarrow \begin{cases} x = \sqrt[3]{u^2 v} \\ y = \sqrt[3]{u v^2} \end{cases}$$

$z = xy = uv.$

$$\begin{cases} \frac{\partial x}{\partial u} = \frac{1}{3}(u^2 v)^{-\frac{2}{3}}\, 2uv & \frac{\partial x}{\partial v} = \frac{1}{3}(u^2 v)^{-\frac{2}{3}}\, u^2 \\ \frac{\partial y}{\partial u} = \frac{1}{3}(uv^2)^{-\frac{2}{3}}\, v^2 & \frac{\partial y}{\partial v} = \frac{1}{3}(uv^2)^{-\frac{2}{3}}\, 2uv \end{cases} \Rightarrow I = \frac{4}{9} - \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$$

$$V = \iint\limits_{\Omega} xy\,dx\,dy = \frac{1}{3}\int\limits_1^2 u\,du \int\limits_1^2 v\,dv = \frac{1}{3}\left(\int\limits_1^2 u\,du\right)^2 = \frac{1}{3}\left(\frac{u^2}{2}\Bigg|_1^2\right)^2 = \frac{1}{3}\left(2 - \frac{1}{2}\right)^2 =$$

$$= \frac{1}{3}\cdot\frac{9}{4} = \frac{3}{4}.$$

Дома №№ 3988, 3993, 3998.1, 3996, 4010, 4014, 4018, 4028.

№3988. Вычислить площадь фигуры, ограниченной кривой:

$$\begin{cases} (x^3+y^3)^2 = x^2+y^2 \\ x \geq 0, \ y \geq 0 \end{cases}$$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \Rightarrow \quad r^6(\cos^3\varphi+\sin^3\varphi)^2 = r^2 \Rightarrow r = \frac{1}{\sqrt{\cos^3\varphi+\sin^3\varphi}} \Rightarrow$$

$$\begin{cases} 0 \leq \varphi \leq \frac{\pi}{2} \\ 0 \leq r \leq \frac{1}{\sqrt{\cos^3\varphi+\sin^3\varphi}} \end{cases} \Rightarrow S = \iint\limits_{\Omega} dx\,dy = \int\limits_0^{\frac{\pi}{2}} d\varphi \int\limits_0^{\frac{1}{\sqrt{\cos^3\varphi+\sin^3\varphi}}} r\,dr =$$

$$= \frac{1}{2}\int\limits_0^{\frac{\pi}{2}} \frac{d\varphi}{\cos^3\varphi+\sin^3\varphi} = \frac{1}{2\sqrt{2}}\int\limits_0^{\frac{\pi}{2}} \frac{d\varphi}{\cos(\varphi-\frac{\pi}{4})(1-\frac{1}{2}\sin 2\varphi)} = \frac{1}{\sqrt{2}}\int\limits_0^{\frac{\pi}{2}} \frac{d\varphi}{\cos(\varphi-\frac{\pi}{4})(2-\cos(2\varphi-\frac{\pi}{2}))} =$$

$$= \frac{1}{\sqrt{2}}\int\limits_0^{\frac{\pi}{2}} \frac{\cos(\varphi-\frac{\pi}{4})\,d\varphi}{(1-\sin^2(\varphi-\frac{\pi}{4}))(1+2\sin^2(\varphi-\frac{\pi}{4}))} = \left| z = \sin(\varphi-\tfrac{\pi}{4}) \right| = \frac{1}{\sqrt{2}}\int\limits_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \frac{dz}{(1-z^2)(1+2z^2)} =$$

$$= \frac{1}{3\sqrt{2}}\int\limits_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \frac{dz}{1-z^2} + \frac{2}{3\sqrt{2}}\int\limits_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \frac{dz}{1+2z^2} = \frac{1}{6\sqrt{2}} \ln\left|\frac{1+z}{1-z}\right| \Bigg|_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} + \frac{\sqrt{2}}{3}\int\limits_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \frac{dz}{1+(\sqrt{2}z)^2} =$$

$$= \left| u = \sqrt{2}z \right| = \frac{1}{3\sqrt{2}} \cdot \ln\frac{\sqrt{2}+1}{\sqrt{2}-1} + \frac{1}{3}\int\limits_{-1}^{1} \frac{du}{1+u^2} = \frac{1}{3\sqrt{2}}\ln(\sqrt{2}+1)^2 + \frac{1}{3}\operatorname{arctg} u \Big|_{-1}^{1} =$$

$$= \frac{\sqrt{2}}{3} \cdot \ln(\sqrt{2}+1) + \frac{1}{3}\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = \frac{\pi}{6} + \frac{\sqrt{2}}{3}\ln(\sqrt{2}+1).$$

№3993 Вычислить площадь, ограниченной кривой:

$$\begin{cases} \left(\frac{x}{a}+\frac{y}{b}\right)^4 = \frac{x^2}{h^2}+\frac{y^2}{k^2} \\ x \geq 0, \ y \geq 0 \end{cases}$$

Замена переменной

$$\begin{cases} x = ar\cos^2\varphi \\ y = br\sin^2\varphi \end{cases} \Rightarrow \quad 0 \leq \varphi \leq \frac{\pi}{2}$$

$$r^4 = \frac{a^2}{h^2}r^2\cos^4\varphi + \frac{b^2}{k^2}r^2\sin^4\varphi \Rightarrow$$

$$\begin{cases} 0 \leq \varphi \leq \frac{\pi}{2} \\ 0 \leq r \leq \sqrt{\frac{a^2}{h^2}\cos^4\varphi + \frac{b^2}{k^2}\sin^4\varphi} \end{cases} \qquad I = 2abr\cos\varphi\sin\varphi$$

$$S = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\sqrt{\frac{a^2}{h^2}\cos^4\varphi + \frac{b^2}{k^2}\sin^4\varphi}} 2abr\cos\varphi\sin\varphi\, dr = ab\int_0^{\frac{\pi}{2}} \cos\varphi\sin\varphi\left(\frac{a^2}{h^2}\cos^4\varphi + \frac{b^2}{k^2}\sin^4\varphi\right)d\varphi =$$

$$= \frac{a^3b}{h^2}\int_0^{\frac{\pi}{2}}\cos^5\varphi\sin\varphi\, d\varphi + \frac{ab^3}{k^2}\int_0^{\frac{\pi}{2}}\sin^5\varphi\cos\varphi\, d\varphi = \left| \begin{matrix} u = \cos\varphi \\ v = \sin\varphi \end{matrix} \right| =$$

$$= \frac{a^3b}{h^2}\int_0^1 u^5 du + \frac{ab^3}{k^2}\int_0^1 v^5 dv = ab\left(\frac{a^2}{h^2} + \frac{b^2}{k^2}\right)\int_0^1 u^5 du = \frac{ab}{6}\left(\frac{a^2}{h^2} + \frac{b^2}{k^2}\right).$$

№3996 Найти площадь фигуры, ограниченной линиями:

$$\begin{cases} x+y = a & y = \alpha x \\ x+y = b & y = \beta x \\ 0 < a < b & 0 < \alpha < \beta \end{cases}$$

Замена переменной:

$$\begin{cases} u = y+x \\ v = \frac{y}{x} \end{cases} \Rightarrow \begin{cases} x = \frac{u}{v+1} \\ y = \frac{uv}{v+1} \end{cases} \Rightarrow \begin{cases} a \le u \le b \\ \alpha \le v \le \beta \end{cases} \Rightarrow$$

$$\begin{cases} \frac{\partial x}{\partial u} = \frac{1}{v+1} & \frac{\partial x}{\partial v} = -\frac{u}{(v+1)^2} \\ \frac{\partial y}{\partial u} = \frac{v}{v+1} & \frac{\partial y}{\partial v} = \frac{u}{(v+1)^2} \end{cases} \Rightarrow I = \frac{u}{(v+1)^3} + \frac{uv}{(v+1)^3} = \frac{u}{(v+1)^2}$$

$$S = \int_a^b du \int_\alpha^\beta \frac{u}{(v+1)^2}dv = \int_a^b u\,du \int_\alpha^\beta \frac{dv}{(v+1)^2} = \frac{u^2}{2}\Big|_a^b \cdot \left(-\frac{1}{v+1}\right)\Big|_\alpha^\beta =$$

$$= \frac{b^2-a^2}{2}\left(-\frac{1}{\beta+1} + \frac{1}{\alpha+1}\right) = \frac{1}{2}\cdot\frac{\beta-\alpha}{(\alpha+1)(\beta+1)}(b^2-a^2).$$

№3998.1 Найти площадь фигуры, ограниченной линиями:

$$\begin{cases} x^2 = ay & x^3 = cy^2 \\ x^2 = by & x^3 = dy^2 \\ 0 < a < b & 0 < c < d \end{cases}$$

Замена переменных:

$$\begin{cases} u = \frac{x^2}{y} \\ v = \frac{x^3}{y^2} \end{cases} \Rightarrow \begin{cases} a \le u \le b \\ c \le v \le d \end{cases} \Rightarrow \begin{cases} x = \frac{u^2}{v} \\ y = \frac{u^3}{v^2} \end{cases} \Rightarrow$$

$$\frac{\partial x}{\partial u} = \frac{2u}{v} \quad \frac{\partial x}{\partial v} = -\frac{u^2}{v^2} \quad \frac{\partial y}{\partial u} = 3\frac{u^2}{v^2} \quad \frac{\partial y}{\partial v} = -2\frac{u^3}{v^3}$$

$$I = -4\frac{u^4}{v^4} + 3\frac{u^4}{v^4} = -\frac{u^4}{v^4} \Rightarrow |I| = \frac{u^4}{v^4}$$

$$S = \iint_{\Omega} dxdy = \int_a^b du \int_c^d \frac{u^4}{v^4} dv = \int_a^b u^4 du \int_c^d \frac{dv}{v^4} = \frac{u^5}{5}\Big|_a^b \cdot \left(-\frac{1}{3}v^{-3}\right)\Big|_c^d =$$

$$= \frac{1}{15}(b^5 - a^5)(c^{-3} - d^{-3}).$$

N4010 Найти объем тела, ограниченного поверхностями:

$$\begin{cases} z = \cos x \cos y \\ z = 0 \\ |x+y| \le \frac{\pi}{2},\ |x-y| \le \frac{\pi}{2} \end{cases}$$

$$V = \iint_{\Omega} \cos x \cos y\, dxdy = \left| \begin{matrix} u = y + x \\ v = y - x \end{matrix} \Rightarrow \begin{matrix} y = \frac{u+v}{2} \\ x = \frac{u-v}{2} \end{matrix} \Rightarrow I = \frac{1}{2} \right| =$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} du \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) dv = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} du \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos v + \cos u)\, dv =$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} du\, (\sin v + \cos u \cdot v)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2 + \pi\cos u)\, du = \frac{1}{4}(2u + \pi \sin u)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= \frac{1}{4}(2\pi + 2\pi) = \pi.$$

N4018 Найти объем тела, ограниченного поверхностями:

$$\begin{cases} z = e^{-(x^2+y^2)} \\ z = 0 \\ x^2 + y^2 = R^2 \end{cases}$$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \Rightarrow \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le R \end{cases} \Rightarrow V = \iint_{\Omega} e^{-(x^2+y^2)} dxdy = \int_0^{2\pi} d\varphi \int_0^R r e^{-r^2} dr =$$

$$= 2\pi \int_0^R r e^{-r^2} dr = -\pi e^{-r^2}\Big|_0^R = \pi(1 - e^{-R^2}).$$

N4028 Найти объем тела, ограниченного поверхностями:

$$\begin{cases} z = x^2 + y^2 & y = \frac{x}{2} \\ xy = a^2 & y = 2x \\ xy = 2a^2 & z = 0 \end{cases}$$

Замена переменных:

$$\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases} \Rightarrow \begin{cases} a^2 \le u \le 2a^2 \\ \frac{1}{2} \le v \le 2 \end{cases} \Rightarrow \begin{cases} x = \sqrt{\frac{u}{v}} \\ y = \sqrt{uv} \end{cases} \Rightarrow z = \frac{u}{v} + uv$$

$$\begin{cases} \frac{\partial x}{\partial u} = \frac{1}{2}\left(\frac{u}{v}\right)^{-\frac{1}{2}} \cdot \frac{1}{v} & \frac{\partial x}{\partial v} = -\frac{1}{2}\left(\frac{u}{v}\right)^{-\frac{1}{2}} \frac{u}{v^2} \\ \frac{\partial y}{\partial u} = \frac{1}{2}(uv)^{-\frac{1}{2}} v & \frac{\partial y}{\partial v} = \frac{1}{2}(uv)^{-\frac{1}{2}} u \end{cases} \Rightarrow$$

$$I = \frac{1}{4} \cdot \frac{\cancel{\sqrt{v}} \cdot \cancel{u}}{\cancel{\sqrt{u}} \cdot v \cdot \cancel{\sqrt{u}} \cdot \cancel{\sqrt{v}}} + \frac{1}{4} \cdot \frac{\cancel{\sqrt{v}} \cdot \cancel{u} \cdot \cancel{v}}{\cancel{\sqrt{u}} \cdot v^2 \cdot \cancel{\sqrt{u}} \cdot \cancel{\sqrt{v}}} = \frac{1}{4v} + \frac{1}{4v} = \frac{1}{2v} \Rightarrow$$

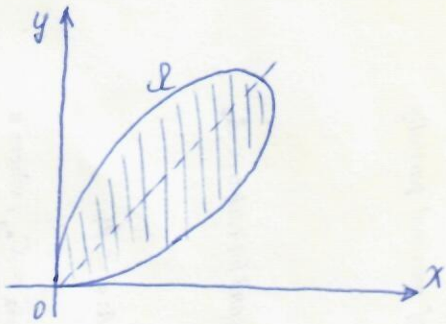
$$V = \int_{a^2}^{2a^2} du \int_{\frac{1}{2}}^{2} \frac{1}{2v}\left(\frac{u}{v} + uv\right) dv = \frac{1}{2}\int_{a^2}^{2a^2} du \int_{\frac{1}{2}}^{2}\left(\frac{u}{v^2} + u\right) dv =$$

$$= \frac{1}{2}\int_{a^2}^{2a^2} du \left(uv - \frac{u}{v}\right)\Big|_{\frac{1}{2}}^{2} = \frac{1}{2}\int_{a^2}^{2a^2}\left(2u - \frac{u}{2} - \frac{u}{2} + 2u\right) du = \frac{1}{2}\int_{a^2}^{2a^2} 3u\, du =$$

$$= \frac{3}{4} u^2 \Big|_{a^2}^{2a^2} = \frac{3}{4}\left(4a^4 - a^4\right) = \frac{9}{4} a^4.$$

№4014. Найти объём тела, ограниченного поверхностями:

$$\begin{cases} z = x + y \\ (x^2+y^2)^2 = 2xy \\ z = 0 \\ x \geq 0,\ y \geq 0 \end{cases}$$



$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \Rightarrow \begin{array}{l} 0 \le \varphi \le \frac{\pi}{2},\ z = \sqrt{2}\, r\cos\left(\varphi - \frac{\pi}{4}\right), \\ r^4 = 2r^2\cos\varphi\sin\varphi \Rightarrow r^2 = \sin 2\varphi \Rightarrow r = \sqrt{\sin 2\varphi} \Rightarrow \end{array}$$

$$\begin{cases} 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le \sqrt{\sin 2\varphi} \end{cases} \Rightarrow V = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\sqrt{\sin 2\varphi}} r^2 \sqrt{2}\cos\left(\varphi - \frac{\pi}{4}\right) dr =$$

$$= \sqrt{2}\int_0^{\frac{\pi}{2}} \cos\left(\varphi - \frac{\pi}{4}\right) d\varphi \int_0^{\sqrt{\sin 2\varphi}} r^2 dr = \frac{\sqrt{2}}{3}\int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}} 2\varphi \cos\left(\varphi - \frac{\pi}{4}\right) d\varphi = \frac{\sqrt{2}}{3}\int_0^{\frac{\pi}{2}} \left(\cos\left(2\varphi - \frac{\pi}{2}\right)\right)^{\frac{3}{2}} \cos\left(\varphi - \frac{\pi}{4}\right) d\varphi =$$

$$= \frac{\sqrt{2}}{3}\int_0^{\frac{\pi}{2}} \left(1 - 2\sin^2\left(\varphi - \frac{\pi}{4}\right)\right)^{\frac{3}{2}} \cos\left(\varphi - \frac{\pi}{4}\right) d\varphi = \left| u = \sqrt{2}\sin\left(\varphi - \frac{\pi}{4}\right) \right| = \frac{1}{3}\int_{-1}^{1} \left(1 - u^2\right)^{\frac{3}{2}} du =$$

$$= \frac{2}{3}\int_0^1 \left(1 - u^2\right)^{\frac{3}{2}} du = \left| \begin{array}{l} u = \sin t \\ du = \cos t\, dt \end{array} \right| = \frac{2}{3}\int_0^{\frac{\pi}{2}} \cos^4 t\, dt = \frac{1}{6}\int_0^{\frac{\pi}{2}} \left(1 + \cos 2t\right)^2 dt =$$

$$= \frac{1}{6}\int_0^{\frac{\pi}{2}} \left(1 + \overset{0}{2\cos 2t} + \frac{1 + \overset{0}{\cos 4t}}{2}\right) dt = \frac{1}{6} \cdot \frac{3}{2} \cdot \frac{\pi}{2} = \frac{\pi}{8}.$$